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THE STRESS INTENSITY FACTOR FOR A CRACK AT THE EDGE OF A LOADED HOLE

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Abstract—The problem considered is that of calculating the stresses near the tip of a radial crack at the edge of a circular hole in an infinite elastic solid when the crack and the hole are loaded in an arbitrary fashion. The problem is reduced to a pair of singular integral equations which are solved numerically. Two particular cases are considered in detail: An infinite sheet uniformly loaded remote from the hole and a localized load acting at the edge of the hole.

1. INTRODUCTION

Consider an infinite elastic solid which, in plane polar coordinates (R, θ), contains a circular hole $0 \le R \le B$, $0 \le \theta \le 2\pi$ and a radial edge crack $B \le R \le C$, $\theta = 0$ (Fig. 1). The problem we wish to investigate is that of determining the stresses in the vicinity of the crack tip when the crack and the hole are loaded in an arbitrary fashion. Via the principle of superposition the problem, which is solved under the assumptions of plane strain, is seen to be equivalent to that in which the hole is traction free and the crack is subject to loads of the form

$$\begin{aligned}\sigma_{\theta\theta}(R,0) = \sigma_{\theta\theta}(R,2\pi) = -p_0 f_1(R/B),\ \ B \le R < C\\ \sigma_{R\theta}(R,0) = \sigma_{R\theta}(R,2\pi) = -p_0 f_2(R/B),\ \ B \le R < C\end{aligned} \tag{1.1}$$

p_0 having the dimensions of stress. We introduce the dimensionless quantities $r = R/B$, $c = C/B$,

$$s_{rr}(r,\theta) = \frac{\sigma_{RR(R,\theta)}}{p_0},\ s_{r\theta} = \frac{\sigma_{R\theta(R,\theta)}}{p_0},\ s_{\theta\theta}(r,\theta) = \frac{\sigma_{\theta\theta(R,\theta)}}{p_0},\ u(r,\theta) = \frac{EU_R(R,\theta)}{p_0(1+\nu)B},\ v(r,\theta) = \frac{EU_\theta(R,\theta)}{p_0(1+\nu)B} \tag{1.2}$$

where E is Young's modulus and ν is Poisson's ratio. In terms of these quantities the problem may be stated as follows:

Find a solution of the dimensionless, plane strain equations of elasticity in the region $\Omega = \{(r,\theta)\colon 1 < r < \infty, 0 < \theta < 2\pi\}$ such that

1. ru, rv and $r^2 s_{ij}$ are continuous functions of (r, θ) in Ω and are $0(1)$ at infinity.
2. $s_{\theta\theta}(r,0) - s_{\theta\theta}(r,2\pi) = 0,\ 1 < r < \infty$
   $s_{r\theta}(r,0) - s_{r\theta}(r,2\pi) = 0,\ 1 < r < \infty$
3. $u(r,0) - u(r,2\pi) = 0,\ c \le r < \infty$
   $v(r,0) - v(r,2\pi) = 0,\ c \le r < \infty$
4. $s_{\theta\theta}(r,0) = -f_1(r),\ 1 \le r < c$
   $s_{r\theta}(r,0) = -f_2(r),\ 1 \le r < c$
5. $s_{rr}(1,\theta) = s_{r\theta}(1,\theta) = 0,\ 0 < \theta < 2\pi$
6. $\lim_{r\to 1+} \frac{\partial}{\partial r}[u(r,0) - u(r,2\pi)] < \infty$
   $\lim_{r\to 1+} \frac{\partial}{\partial r}[v(r,0) - v(r,2\pi)] < \infty.$

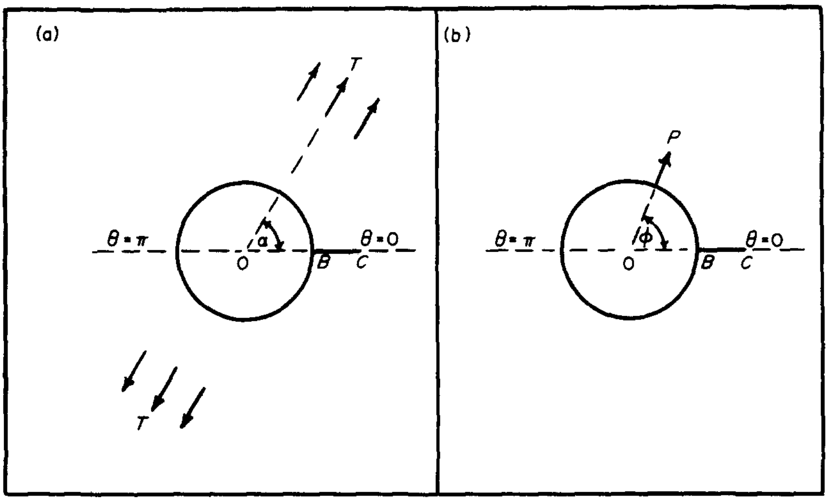

Fig. 1. (a) Cracked hole in an uniaxial tensile stress field T. (b) cracked hole with a force P acting at the perimeter.

2. REDUCTION OF THE PROBLEM TO INTEGRAL EQUATIONS

The problem under consideration may be solved by making use of a Mellin transform technique akin to that discussed by Tweed and Rooke[2]. Alternatively, the problem may be solved in terms of continuous distributions of edge dislocations by making use of the results derived in Dunders and Sendeckyi[3]. In either case it is not difficult to show that the solution is such that

$$u(r,o)-u(r,2\pi)=-4(1-\nu)\int_r^c \frac{P_2(t)\,dt}{\sqrt{[(c-t)(t-1)]}},\quad 1<r<c \tag{2.1}$$

and

$$v(r,0)-v(r,2\pi)=-4(1-\nu)\int_r^c \frac{P_1(t)\,dt}{\sqrt{[(c-t)(t-1)]}},\quad 1<r<c, \tag{2.2}$$

where the quantities $P_i(t)$ $i=1,2$ must satisfy the singular integral equations.

$$\frac{1}{\pi}\int_1^c \frac{P_i(t)}{\sqrt{[(c-t)(t-1)]}}\left\{\frac{1}{t-r}+K_i(r,t)\right\}dt=-f_i(r),\quad 1<r<c \tag{2.3}$$

with subsidiary conditions

$$P_i(1)=0 \qquad 1=1,2, \tag{2.4}$$

and kernel functions

$$K_1(r,t)=\frac{(1-t^2)^2}{t(1-rt)^3}-\frac{t(1-t^2)}{(1-rt)^2}-\frac{t}{(1-rt)}+\frac{1-t^2}{r^2t} \tag{2.5}$$

and

$$K_2(r,t)=\frac{(1-t^2)^2}{t(1-rt)^3}-\frac{(1-t^2)}{t(1-rt)^2}-\frac{t}{(1-rt)}. \tag{2.6}$$

3. THE STRESS INTENSITY FACTORS AND CRACK ENERGY

The opening and sliding mode stress intensity factors k_1 and k_2 respectively, and the crack formation energy W are defined by the equations

$$k_1 = -\frac{E}{4(1-\nu^2)} \lim_{R\to C_-} \sqrt{[2(C-R)]} \frac{\partial}{\partial R}[u_\theta(R,0) - u_\theta(R,2\pi)], \tag{3.1}$$

$$k_2 = -\frac{E}{4(1-\nu^2)} \lim_{R\to C_-} \sqrt{[2(C-R)]} \frac{\partial}{\partial R}[u_R(R,0) - u_R(R,2\pi)], \tag{3.2}$$

and

$$W = \frac{1}{2}\int_B^C p_0 f_1(R/B)[u_\theta(R,0) - u_\theta(R,2\pi)]\,\mathrm{d}R + \frac{1}{2}\int_B^C p_0 f_2(R/B)[u_R(R,0) - u_R(R,2\pi)]\,\mathrm{d}R. \tag{3.3}$$

It is well known[4] that the stress intensity factor k_0 and the crack formation energy W_0 of a crack of length $2(C-B)$ in an infinite elastic solid which is subject to a uniform all round tension p_0 are given by $k_0 = p_0\sqrt{(C-B)}$ and $W_0 = \pi(1-\nu^2)p_0^2(C-B)^2/E$ respectively. Therefore, from (1.2), (2.1) and (2.2) we have

$$\frac{k_1}{k_0} = \frac{-\sqrt{(2)}}{(c-1)} P_1(c) \tag{3.4}$$

$$\frac{k_2}{k_0} = \frac{-\sqrt{(2)}}{(c-1)} P_2(c) \tag{3.5}$$

and

$$\frac{W}{W_0} = -\frac{2}{\pi(c-1)^2}\int_1^c \frac{P_1(t)}{\sqrt{[(c-t)(t-1)]}}\int_1^t f_1(r)\,\mathrm{d}r\,\mathrm{d}t - \frac{2}{\pi(c-1)^2}\int_1^c \frac{P_2(t)}{\sqrt{[(c-t)(t-1)]}}\int_1^t f_2(r)\,\mathrm{d}r\,\mathrm{d}t. \tag{3.6}$$

4. UNIAXIAL TENSION

The first special case we consider is that in which an unixial tensile stress T acts remote from the crack in the direction $\theta = \alpha$ (see Fig. 1a). In this case we have[5] that $p_0 = T$, $f_1(r) = \frac{1}{2}(1 + r^{-2}) - \frac{1}{2}(1 + 3r^{-4})\cos 2\alpha$ and $f_2(r) = \frac{1}{2}(1 + 2r^{-2} - 3r^{-4})\sin 2\alpha$. Following Erdogan and Gupta[6] we set $x_k = \cos[(2k-1)\pi/2n]$, $t_k = \frac{1}{2}(c-1)x_k + \frac{1}{2}(c+1)$ and $r_j = \frac{1}{2}(c-1)\cos(j\pi/n) + \frac{1}{2}(c+1)$, $k = 1,2,\ldots,n$; $j = 1,2,\ldots,(n-1)$, and then replace the eqns (2.3), (2.4) by the linear algebraic system

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} P_i(t_k)\left\{\frac{1}{t_k - r_j} + K_i(r_j, t_k)\right\} &= -f_i(r_j)\\ \frac{1}{n}\sum_{k=1}^{n} (-)^k P_i(t_k)\left(\frac{1-x_k}{1+x_k}\right)^{1/2} &= 0\end{aligned} \tag{4.1}$$

$j = 1,2,3,\ldots,(n-1)$; $i = 1,2$. Having solved the system (4.1) for the $P_i(t_k)$ we calculate k_i/k_0 from the Chebyshev–Lagrange interpolation formula

$$\frac{k_i}{k_0} = \frac{\sqrt{(2)}}{n(c-1)}\sum_{k=1}^{n} (-)^k \left(\frac{1+x_k}{1-x_k}\right)^{1/2} P_i(t_k),\ i = 1,2 \tag{4.2}$$

and W/W_0 from the Gaussian quadrature formula

$$\begin{aligned}\frac{W}{W_0} = &-\frac{1}{n(c-1)^2}\sum_{k=1}^{n} P_i(t_k)\{(t_k - t_k^{-1}) - (t_k - t_k^{-3})\cos 2\alpha\} - \frac{1}{n(c-1)^2}\sum_{k=1}^{n} P_i(t_k)\\ &\times (t_k - 2t_k^{-1} + t_k^{-3})\sin 2\alpha.\end{aligned} \tag{4.3}$$

The stress intensity factors are shown as plots of k_1/k_0 and k_2/k_0 *vs* $(C-B)/B$ for various values of α, from 0° to 90°, in Fig. 2. The values of $k_2(\alpha)$ are symmetrical about $\alpha = 45°$. The results for $\alpha = 0°$ and 90° are identical to those previously obtained for a crack at the edge of a circular hole in a sheet under biaxial tension[2]. The crack formation energy is plotted as W/W_0 vs $(C-B)/B$ in Fig. 3 for various α; the results for $\alpha = 0°$ and 90° agree with previous work[2].

Results for this case have been obtained previously by Hsu[7] using conformal mapping techniques. The procedure described in this paper is more efficient in computing requirements: the integral eqn (4.1) was reduced to 24 simultaneous linear equations to give results accurate to about 0.1% whereas the conformal mapping technique requires between 40 and 140 equations to give results of equivalent accuracy. A second advantage of the present technique is that it suffers no loss in accuracy for short cracks; the edge crack limit is reproduced as $C/B \to 1+$. The mapping technique becomes increasingly inaccurate as the cracks become shorter and no reliable results were obtained by Hsu[7] for $C/B < 1.1$. Reliable stress intensity factors for short cracks are required for many important technological applications, in particular they are required in the fulfilment of damage-tolerant design criteria in the development of aircraft structures.

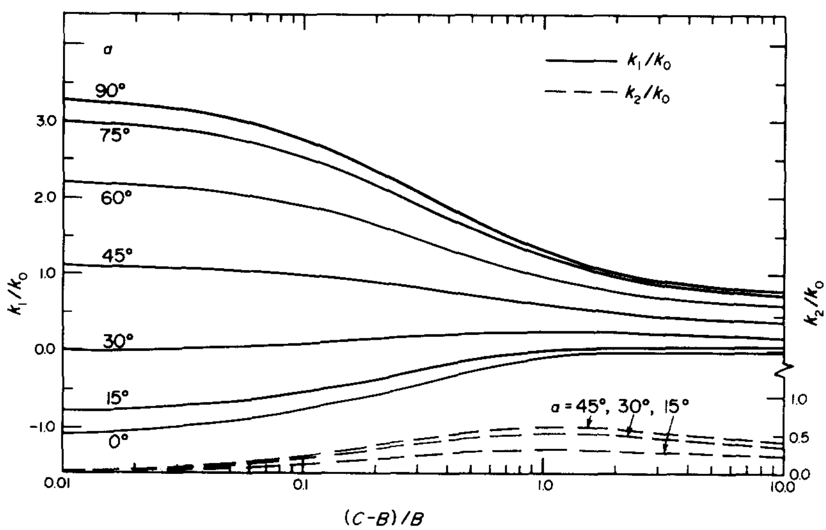

Fig. 2. Stress intensity factors for a crack at the edge of a circular hole: uniaxial tensile stress remote from the crack.

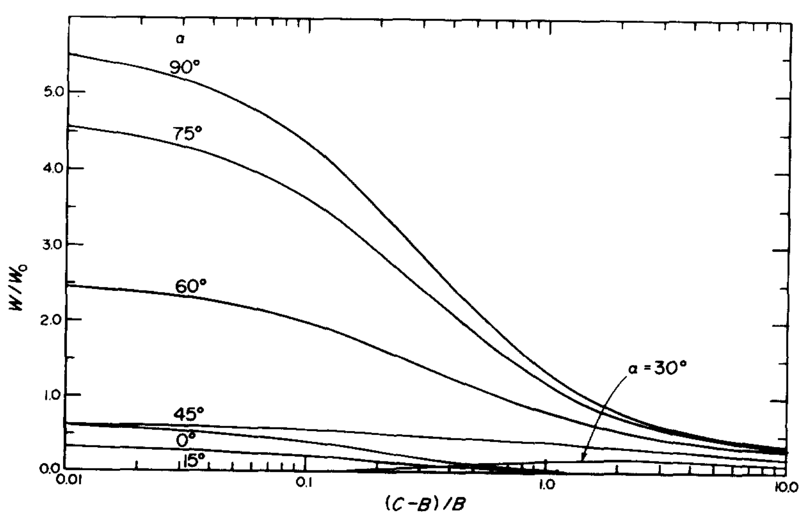

Fig. 3. Crack formation energy for a crack at the edge of a circular hole: uniaxial tensile stress remote from the crack.

5. POINT LOAD ACTING AT THE CIRCUMFERENCE OF THE HOLE

The second case we consider is that in which a force of magnitude F acts in the radial direction at the point (B, ϕ) on the circumstance of the hole. Green and Zerna[8] have shown that in this case $p_0 = F/2B$, $f_1/(r) = g_1(r) + \alpha g_2(r)$ and $f_2(r) = g_3(r) + \alpha g_4(r)$ where $\alpha = (3-4\nu)/2(1-\nu)$,

$$g_1(r) = \frac{1}{\pi}\left[\frac{1}{r^2} + \frac{4}{X^2} - \frac{2r\cos\phi}{X^2} - \frac{(1+4r^2-r^4)\cos\phi}{rX^4}\right], \tag{5.1}$$

$$g_2(r) = \frac{1+r^2}{\pi r^3}\cos\phi, \tag{5.2}$$

$$g_3(r) = \frac{(1-r^2)\sin\phi}{\pi r X^4}, \tag{5.3}$$

$$g_4(r) = \frac{1-r^2}{\pi r^3}\sin\phi, \tag{5.4}$$

and

$$X^2 = 1 - 2r\cos\phi + r^2. \tag{5.5}$$

If we let

$$P_1(r) = Q_1(r) + \alpha Q_2(r), \tag{5.6}$$

$$P_2(r) = Q_3(r) + \alpha Q_4(r), \tag{5.7}$$

$$M_1(r,t) = M_2(r,t) = (t-r)^{-1} + K_1(r,t), \tag{5.8}$$

and

$$M_3(r,t) = M_4(r,t) = (t-r)^{-1} + K_2(r,t), \tag{5.9}$$

then (2.24) and (12.29) are equivalent to the equations

$$\frac{1}{\pi}\int_1^c \frac{Q_i(t)M_i(r,t)\,dt}{\sqrt{[(c-t)(t-1)]}} = -g_i(r),\ 1<r<c$$

$$Q_i(1) = 0 \tag{5.10}$$

$i = 1, 2, 3, 4$. Therefore, by the method of Erdogan and Gupta[6], we are led to the linear algebraic system

$$\frac{1}{n}\sum_{k=1}^{n} Q_i(t_k)M_i(r_j, t_k) = -g_i(r_j)$$

$$\frac{1}{n}\sum_{k=1}^{n} (-)^k Q_i(t_k)\left(\frac{1-x_k}{1+x_k}\right)^{1/2} = 0 \tag{5.11}$$

$j = 1, 2, 3, \ldots, (n-1)$; $i = 1,2,3,4$.

The dimensionless stress intensity factors and crack formation energy may now be calculated from the formulae

$$\frac{k_1}{k_0} = \frac{k^{(1)}}{k_0} + \alpha\frac{k^{(2)}}{k_0},\quad \frac{k_2}{k_0} = \frac{k^{(3)}}{k_0} + \alpha\frac{k^{(4)}}{k_0} \tag{5.12}$$

and

$$\frac{W}{W_0} = W_{11} + W_{33} + \alpha(W_{12} + W_{21} + W_{34} + W_{43}) + \alpha^2(W_{22} + W_{44}), \tag{5.13}$$

where

$$\frac{k^{(1)}}{k_0} = \frac{\sqrt{(2)}}{n(c-1)} \sum_{k=1}^{n} (-)^k \left(\frac{1+x_k}{1-x_k}\right)^{1/2} Q_i(t_k), \tag{5.14}$$

$$W_{ij} = -\frac{2}{n(c-1)^2} \sum_{k=1}^{n} Q_i(t_k) G_j(t_k), \tag{5.15}$$

and

$$G_i(t) = \int_1^t g_i(r)\,\mathrm{d}r. \tag{5.16}$$

$i, j = 1, 2, 3, 4.$

Explicit expressions for the functions $G_i(t)$ are given in the Appendix. From Betti's theorem it follows that $W_{12} = W_{21}$ and $W_{34} = W_{43'}$ this provides a useful check on the numerical procedure outlined above.

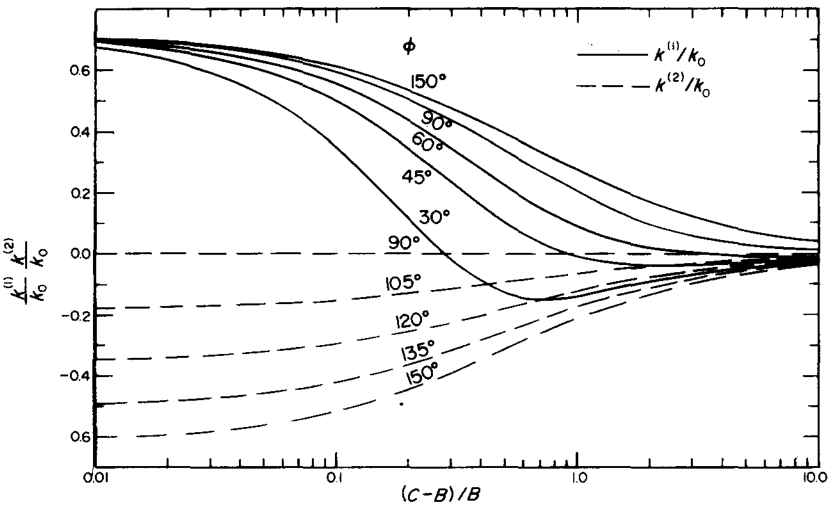

Fig. 4. Components of the opening mode stress intensity factor for a crack at the edge of a point-loaded circular hole.

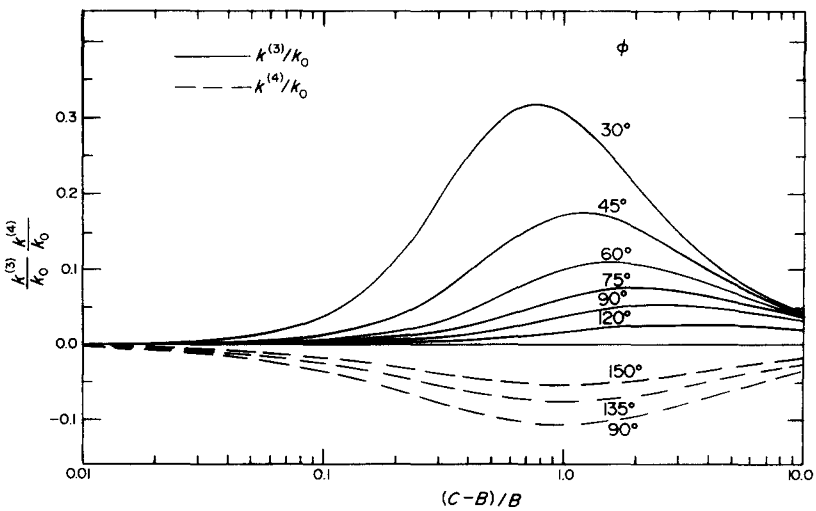

Fig. 5. Components of the sliding mode stress intensity factors for a crack at the edge of a point-loaded circular hole.

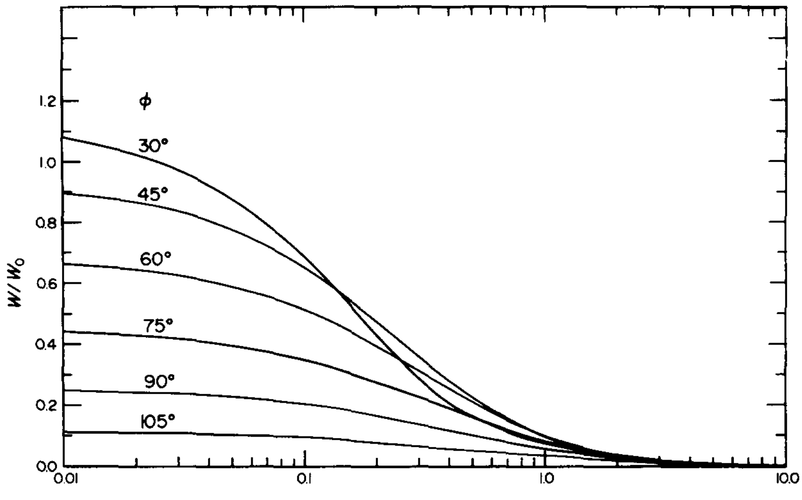

Fig. 6. Crack formation energy for a crack at the edge of a point-loaded circular hole.

The two components $k^{(1)}$ and $k^{(2)}$ of the opening mode stress intensity factor k_1 are shown plotted as $k^{(1)}/k_0$ and $k^{(2)}/k_0$ vs $(C-B)/B$ in Fig. 4 for various values of ϕ ($30° \le \phi \le 150°$). The results for $k^{(1)}$ at $\phi = 90°$ agree with previous work[2]; $k^{(2)}$ obeys the relation $k^{(2)}(180° - \phi) = -k^{(2)}(\phi)$. Figure 5 shows the two components $k^{(3)}$ and $k^{(4)}$ of the sliding mode stress intensity factor k_2 plotted as $k^{(3)}/k_0$ and $k^{(4)}/k_0$ vs $(C-B)/B$ for various values of ϕ. The component $k^{(4)}$ obeys the relation $k^{(4)}(180° - \phi) = k^{(4)}(\phi)$. The crack formation energy is plotted as W/W_0 vs $(C-B)/B$ in Fig. 6.

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APPENDIX

The functions $G_i(t)$

The functions $G_i(t)$ are defined by the equation

$$G_i(t) = \int_1^t g_i(r)\,dr$$

$i = 1, 2, 3, 4$ where the $g_i(r)$ are given by (5.1) through (5.5). Therefore, if we let

$$I_{m,n} = \int_1^t \frac{r^m}{x^{2n}}\,dr$$

where $x^2 = 1 - 2r\cos\phi + r^2$, we find that

$$G_1 = \frac{1}{\pi}[I_{-2,0} + 4I_{0,2}] - \frac{\cos\phi}{\pi}[2I_{1,1} + I_{-1,2} + 4I_{1,2} - I_{3,2}]$$

$$G_2 = \frac{1}{\pi}[I_{-3,0} + I_{-1,0}]\cos\phi$$

$$G_3 = \frac{1}{\pi}[I_{-1,2} - 2I_{1,2} + I_{3,2}]\sin\phi$$

and

$$G_4 = \frac{1}{\pi}[I_{-3,0} - I_{-1,0}]\sin\phi.$$

The integrals $I_{m,n}$ are described in detail in Gradshteyn and Ryshik[9]. In particular, we have

$$I_{-3,0} = \frac{1}{2}(1 - t^{-2}),$$

$$I_{-2,0} = 1 - t^{-1},$$

$$I_{-1,0} = \log t,$$

$$I_{0,1} = \frac{2}{\sqrt{(\Delta)}}\left\{\tan^{-1}\left(\frac{2t+b}{\sqrt{(\Delta)}}\right) - \tan^{-1}\left(\frac{2+b}{\sqrt{(\Delta)}}\right)\right\},$$

$$I_{1,1} = \frac{1}{2}\log(R_t/R_1) - \frac{1}{2}bI_{0,1}$$

$$I_{0,2} = \frac{b}{\Delta}\left(\frac{1}{R_t} - \frac{1}{R_1}\right) + \frac{2}{\Delta}\left(\frac{t}{R_t} - \frac{1}{R_1}\right) + \frac{2}{\Delta}I_{0,1},$$

$$I_{-1,2} = \frac{1}{2}\log\left(\frac{t^2R_1}{R_t}\right) + \frac{1}{2}\left(1 - \frac{b^2}{\Delta}\right)\left(\frac{1}{R_t} - \frac{1}{R_1}\right) - \frac{b}{\Delta}\left(\frac{t}{R_t} - \frac{1}{R_1}\right) - \frac{b}{2}\left(1 + \frac{2}{\Delta}\right)I_{0,1}$$

$$I_{1,2} = -\frac{2}{\Delta}\left(\frac{1}{R_t} - \frac{1}{R_1}\right) - \frac{b}{\Delta}\left(\frac{t}{R_t} - \frac{1}{R_1}\right) - \frac{b}{\Delta}I_{0,1}$$

and

$$I_{3,2} = \frac{1}{2}\log\left(\frac{R_t}{R_1}\right) + \frac{(2-b^2)}{\Delta}\left(\frac{1}{R_t} - \frac{1}{R_1}\right) + \frac{b(3-b^2)}{\Delta}\left(\frac{t}{R_t} - \frac{1}{R_1}\right) - \frac{b}{2\Delta}(6 - b^2)I_{0,1}.$$

where $b = -2\cos\phi$, $\Delta = 4\sin^2\phi$, $R_t = 1 - 2t\cos\phi + t^2$ and $R_1 = 2(1 - \cos\phi)$.